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Analysis of a multistate control problem related to food technology ☆

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Abstract

This paper is concerned with an optimal control problem related to the determination of an optimal profile for the steam temperature into the autoclave along the processing of canned foods. The problem studies a system coupling the evolution Navier–Stokes equations with the heat transfer equation by natural convection (the so-called Boussinesq equations), and with the microorganisms removal equation. The essential difficulties in the study of this multistate control problem arise from the lack of uniqueness for the solution of the state system. Here we obtain—after a careful analysis of the problem mathematical formulation—the uniqueness of part of the state, and the existence of optimal solutions.
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1. Introduction

The most usual technique of thermal processing for long time food conservation is canning, in which the foods are previously packed and then sterilized. With this method, the containers (cans, bottles, pouches...) are heated inside an autoclave with steam during a time interval, long enough to reduce the pathogenous microorganisms concentrations down to suitable levels. Unfortunately, this technique usually makes the foods to be thermally overprocessed, because of the preventive measures usually observed by food industry. This excessive security factor in thermal processing leads to an unnecessary increase in energy costs, a nutrient degradation and a deterioration of the organoleptic properties: color, taste, smell... Mathematical modelling and optimal control, applied to this industrial problem, can offer a very interesting tool for analyzing these processes, increasing their efficiency and improving the technical design of the related equipment.

According to the heat transfer mechanism, canned foods are divided into conduction heated and convection heated. Most of the mathematical literature in this area has been devoted to conduction heated products. However, this analysis is only suitable for heating of "solid" foods, and not for "liquid" foods, where the heat transfer inside the containers occurs by natural convection. Natural convection in enclosures has been analyzed for such diverse applications as nuclear reactors, petrochemical industries or alimentary processes. In the mathematical literature related to food industry so much finite elements as finite differences methods have been applied with different rates of success in the numerical resolution of the model (cf., for instance, Stevens [15], Engelman and Sani [9], Datta [8], Kumar et al. [12], and the references therein). From a theoretical point of view, the first study on the modelling and control of sterilization processes where heat transfer occurs by natural convection has been developed by Alvarez-Vázquez and Martínez [3]. (A more complex theory for "viscous" canned foods, with temperature-dependent viscosity has been recently derived by Alvarez-Vázquez et al. in [2].)

In all of these works an artificial term involving turbulence minimization is included in the cost function, following an idea of Abergel and Temam [1] and Casas [7], in such a way that the solution of the state system be unique and, consequently, the relation control-state be single-valued. Therefore, the derivation of the optimality conditions for the optimal solutions will be an obvious (although laborious) computation. Nevertheless, for the sake of reliability, in our problem we will not take into account this turbulence term. So, we cannot expect uniqueness for the solution of the state system (since the weak solutions of the evolution three-dimensional Navier–Stokes equations are involved). Thus, we will have to deal with a so-called multistate optimal control problem, where the existence of optimal controls is not a direct result. In recent times several results have been achieved for non-well-posed optimal control problems related to the Boussinesq equations (see, for instance, Wang [18] or Li [14]), but all of them under very strong and non-realistic regularity assumptions on the velocity of the type $\vec{y} \in L^2(0,T;H^2(\Omega)^3) \cap W^{1,2}(0,T;L^2(\Omega)^3)$.

In this paper we study, from a mathematical point of view, an optimal control problem related to the determination of a suitable profile for the steam temperature in the autoclaves during the processing time. The problem deals with a system of partial differential equations coupling the Navier–Stokes equations with the heat transfer equation by natural convection (the so-called Boussinesq system), coupled with the convection–reaction–diffusion equation for the evolution of the microorganisms concentration. This problem will be subject to constraints on the control (steam temperature) and on the state (microorganisms concentration) which the process must satisfy.

Thus, in Section 2 we deeply analyze the problem: we present a rigorous mathematical formulation of the industrial problem as a boundary optimal control problem with control and state constraints, and we study several topics related to the existence of solution for the state system. In Section 3 we prove an abstract result for parabolic equations, which will allow us to derive the uniqueness of part of the state (the temperature component), under very weak regularity assumptions on the velocity (mainly, $\vec{y} \in L^2(0, T; L^3(\Omega)^3)$. In the proof of this new result, the lack of integrability of the velocity makes useless the standard techniques, which compel us to employ more sophisticated arguments. Finally, in Section 4 we obtain the existence of optimal solutions for the control problem. In this case the main difficulties arise as a consequence of the complexity of the state system, of the existence of constraints on control and state, and, essentially, of the lack of uniqueness for the solution of the state system.

As already mentioned in above paragraphs, the food sterilization problem is essentially a boundary control problem and not a distributed control problem, because we cannot act on the food inside the container (since this is a hermetically closed recipient) but only on its boundary. Moreover, from a technological viewpoint, the only reasonable control seems to be the steam temperature in the autoclave, because the velocity and the microorganism concentration are necessarily null on the boundary, due to the sterilization process. Nevertheless, within an abstract framework—not related to our food technology problem—there exist several works in the mathematical literature with more than one control (mainly, the velocity and the temperature) acting, not only on the boundary, but also in the whole domain. Among these works it is worthwhile mentioning those of Fattorini and Sritharan [10], Trenchea [17] or Li [14].

Finally, some words should be said on the necessary optimality conditions. For any optimal control such that the corresponding velocity $\vec{y} \in L^4(0, T; L^8(\Omega)^3)$ (i.e. $\vec{y}$ is a strong solution of the Navier–Stokes equations and, consequently, is unique) optimality conditions can be easily obtained in the standard way following, for instance, the techniques of Casas [7]. If we assume stronger hypothesis on the corresponding velocity (as given, for instance, in Li [14]), optimality conditions can be also obtained by passing to the limit in a sequence of approximation problems. However, in our case of non-smooth velocity, the determination of optimality conditions is, as far as we know, still an open problem.

2. Setting of the optimal control problem

The main goal for the design of thermal processes in the sterilization of canned liquid foods is finding the optimal steam temperature in the autoclave in such a way that the mean concentration of microorganisms along the sterilization process be lower than a fixed threshold, given by the sanitary directives ruling the pathogenous microorganisms removal.

This problem can be formulated as a boundary optimal control problem, where the state system is given by the evolution Navier–Stokes equations coupled with the heat equation for the fluid inside the can (Boussinesq equations), and coupled with the convection–diffusion–reaction equation modelling the removal of microorganisms by effect of the temperature. The control (steam temperature inside the autoclave) is incorporated through a Neumann condition for the heat equation along a part of the boundary.

In this problem, the cost function is related to the minimization of the energy cost (in terms of the steam temperature). Moreover, besides the constraints on control (related to technological bounds for the steam temperature), we also have to include constraints on state (mean microorganisms concentration lower than a given threshold), which increases the main difficulty in order to study the existence of optimal solutions: the non-uniqueness of the solution of the state system.

An abstract related problem has been previously studied, for instance, by Casas [7], where the cost function also includes a term related to the norm of the vorticity in suitable functional spaces. This minimization of the turbulence within the flow implies that the Navier–Stokes equations have a so-called strong solution, which is unique, allowing the author to avoid our main difficulty. But this is not our case, because we have no reason to introduce the vorticity term and, consequently, we cannot obtain the uniqueness of the weak solution of the Navier–Stokes equations.

For the mathematical formulation of the industrial problem, the fluid is supposed to occupy a physical domain $\Omega \subset \mathbb{R}^3$ along the time interval $(0, T)$. We assume that Ω is bounded and its boundary Γ is smooth enough and divided in two non-empty parts: $\Gamma^0 \cup \Gamma^1 = \Gamma$, $\Gamma^0 \cap \Gamma^1 = \emptyset$. We also introduce the following functional spaces:

$$
\begin{aligned}
Y &= \{\vec{y} \in H^1(\Omega)^3 \colon \nabla \cdot \vec{y} = 0\}, \\
Y_0 &= \{\vec{y} \in H_0^1(\Omega)^3 \colon \nabla \cdot \vec{y} = 0\}, \\
W_0 &= \{\xi \in H^1(\Omega) \colon \xi|_{\Gamma_0} = 0\}, \\
W_1 &= \{\xi \in H^1(\Omega) \colon \xi|_{\Gamma_1} = 0\}, \\
H &= \overline{Y_0}^{L^2(\Omega)^3}.
\end{aligned}
$$

Finally, for an arbitrary Hilbert space V with inner product $(.,.)_V$ we consider:

$$
\mathcal{C}_w([0,T]; V) = \{u : [0,T] \to V \text{ such that } (u(.), v)_V \in \mathcal{C}([0,T]),\ \forall v \in V\}.
$$

The state system for this problem will be the Navier–Stokes equations and the heat equation (the so-called Boussinesq system), coupled with the convection–diffusion–reaction equation modelling the removal of microorganisms by effect of the temperature:

$$
\left\{
\begin{array}{l}
\dfrac{\partial \vec{y}}{\partial t} - \nu \Delta \vec{y} + (\vec{y} \cdot \nabla)\vec{y} + \nabla \pi = \vec{f} + \vec{\beta}\theta \quad \text{in } \Omega_T, \\
\nabla \cdot \vec{y} = 0 \quad \text{in } \Omega_T, \\
\dfrac{\partial \theta}{\partial t} - k \Delta \theta + \vec{y} \cdot \nabla \theta = g \quad \text{in } \Omega_T, \\
\dfrac{\partial c}{\partial t} - \alpha \Delta c + \vec{y} \cdot \nabla c + r(\theta) c = 0 \quad \text{in } \Omega_T, \\
\vec{y} = 0 \quad \text{on } \Gamma_T, \\
\theta = 0 \quad \text{on } \Gamma_T^0, \\
\dfrac{\partial \theta}{\partial n} = u \quad \text{on } \Gamma_T^1, \\
\dfrac{\partial c}{\partial n} = 0 \quad \text{on } \Gamma_T^0, \\
c = 0 \quad \text{on } \Gamma_T^1, \\
\vec{y}(0) = \vec{\Phi}_0 \quad \text{in } \Omega, \\
\theta(0) = \tau_0 \quad \text{in } \Omega, \\
c(0) = \Psi_0 \quad \text{in } \Omega,
\end{array}
\right.
\tag{1}
$$

where

- $\vec{y}$ represents the velocity of the fluid inside the can,
- θ represents its temperature,
- $c \geqslant 0$ represents the concentration of microorganisms inside the fluid,
- $\Omega_T = \Omega \times (0,T)$, $\Gamma_T = \Gamma \times (0,T)$, and $\Gamma_T^i = \Gamma^i \times (0,T)$, for $i = 0, 1$,
- the physical coefficients $\nu, k, \alpha > 0$,
- the body forces $\vec{f} \in L^2(0,T; L^2(\Omega)^3)$, $g \in L^2(0,T; L^2(\Omega))$, and $\vec{\beta} \in L^\infty(\Omega_T)$,
- the reaction function $r : L^2(0,T; L^2(\Omega)) \to L^\infty(\Omega_T)$ is a non-negative mapping of class $\mathcal{C}^1$ such that r and its differential Dr are bounded (cf. a realistic expression of r, for instance, in [3]),
- the Neumann condition $u \in L^2(\Gamma_T^1)$ for the temperature represents the control of our problem: the steam temperature in the sterilization retort,
- the initial conditions $\vec{\Phi}_0 \in L^2(\Omega)^3$ and $\tau_0, \Psi_0 \in L^2(\Omega)$.

Before analyzing the problem, let us introduce some notation and previous remarks. We consider the following mappings:

$$a : H^1(\Omega)^3 \times H^1(\Omega)^3 \to \mathbb{R}, \qquad a(\vec{z}^1, \vec{z}^2) = \sum_{j=1}^{3} \int_\Omega \nabla z_j^1 \cdot z_j^2 \, dx,$$

$$b : H^1(\Omega)^3 \times H^1(\Omega)^3 \times H^1(\Omega)^3 \to \mathbb{R}, \qquad b(\vec{z}^1, \vec{z}^2, \vec{z}^3) = \sum_{i,j=1}^{3} \int_\Omega z_i^1 \partial_{x_i} z_j^2 z_j^3 \, dx,$$

$$a_0 : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \qquad a_0(\xi^1, \xi^2) = \int_\Omega \nabla \xi^1 \cdot \nabla \xi^2 \, dx,$$

$$b_0 : H^1(\Omega)^3 \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \qquad b_0(\vec{z}, \xi^1, \xi^2) = \int_\Omega (\vec{z} \cdot \nabla \xi^1) \xi^2 \, dx.$$

Remark 1. It is easy to prove that Y, Y_0 and W_i, $i = 0, 1$, are separable Hilbert spaces when endowed with the norms induced by the following inner products:

$$(\vec{y}, \vec{z})_Y = (\vec{y}, \vec{z})_{L^2(\Omega)^3} + a(\vec{y}, \vec{z}),$$
$$(\vec{y}, \vec{z})_{Y_0} = a(\vec{y}, \vec{z}),$$
$$(\theta, \xi)_{W_i} = a_0(\theta, \xi), \quad i = 0, 1.$$

Remark 2. It is also well known (cf., for instance, Temam [16]) that the trilinear form b is well defined and continuous if considered in $L^4(\Omega)^3 \times H^1(\Omega)^3 \times L^4(\Omega)^3$, in $L^4(\Omega)^3 \times W^{1,4}(\Omega)^3 \times L^2(\Omega)^3$, or in $L^3(\Omega)^3 \times H^1(\Omega)^3 \times H^1(\Omega)^3$. (A similar result is true for the trilinear form b_0.)

Another important property of b is that, for all $\vec{y} \in Y_0$, we have that $b(\vec{y}, \vec{z}^1, \vec{z}^2) = -b(\vec{y}, \vec{z}^2, \vec{z}^1)$, for all $\vec{z}^1, \vec{z}^2 \in H_0^1(\Omega)^3$, from which we can deduce that $b(\vec{y}, \vec{z}, \vec{z}) = 0$, for all $\vec{z} \in H_0^1(\Omega)^3$ and for all $\vec{y} \in Y_0$. (In an analogous way, $b_0(\vec{y}, \xi, \xi) = 0$, for all $\xi \in H^1(\Omega)$ and for all $\vec{y} \in Y_0$.)

With previous notations, the state system (1) can be formulated in the following variational form: We look for $(\vec{y}, \theta, c) \in L^2(0,T; Y_0) \times L^2(0,T; W_0) \times L^2(0,T; W_1)$ satisfying

$$\begin{cases} \dfrac{d}{dt}\big(\vec{y}(t), \vec{\Psi}\big)_{L^2(\Omega)^3} + \nu a\big(\vec{y}(t), \vec{\Psi}\big) + b\big(\vec{y}(t), \vec{y}(t), \vec{\Psi}\big) \\ \quad = \big(\vec{f}(t) + \vec{\beta}(t)\theta(t), \vec{\Psi}\big)_{L^2(\Omega)^3}, \quad \forall \vec{\Psi} \in Y_0, \text{ in } \mathcal{D}'(0,T), \\ \dfrac{d}{dt}\big(\theta(t), \xi\big)_{L^2(\Omega)} + k a_0\big(\theta(t), \xi\big) + b_0\big(\vec{y}(t), \theta(t), \xi\big) \\ \quad = \big(g(t), \xi\big)_{L^2(\Omega)} + \big(u(t), \xi\big)_{L^2(\Gamma_T^1)}, \quad \forall \xi \in W_0, \text{ in } \mathcal{D}'(0,T), \\ \dfrac{d}{dt}\big(c(t), \eta\big)_{L^2(\Omega)} + \alpha a_0\big(c(t), \eta\big) + b_0\big(\vec{y}(t), c(t), \eta\big) + \big(r(\theta)(t)c(t), \eta\big)_{L^2(\Omega)} \\ \quad = 0, \quad \forall \eta \in W_1, \text{ in } \mathcal{D}'(0,T), \\ \vec{y}(0) = \vec{\Phi}_0, \quad \theta(0) = \tau_0, \quad c(0) = \Psi_0. \end{cases} \tag{2}$$

Remark 3. If we consider the operators:

$$A : \vec{y} \in Y_0 \to A(\vec{y}) \in Y_0' \quad \text{such that} \quad \big\langle A(\vec{y}), \vec{z}\big\rangle = a(\vec{y}, \vec{z}), \quad \forall \vec{z} \in Y_0,$$

$$B : \vec{y} \in Y_0 \to B(\vec{y}) \in Y_0' \quad \text{such that} \quad \big\langle B(\vec{y}), \vec{z}\big\rangle = b(\vec{y}, \vec{y}, \vec{z}), \quad \forall \vec{z} \in Y_0,$$

$$A_0 : \theta \in W_0 \to A_0(\theta) \in W_0' \quad \text{such that} \quad \big\langle A_0(\theta), \xi\big\rangle = a_0(\theta, \xi), \quad \forall \xi \in W_0,$$

$$B_0 : (\vec{y}, \theta) \in Y_0 \times W_0 \to B_0(\vec{y}, \theta) \in W_0' \quad \text{such that}$$
$$\quad \big\langle B_0(\vec{y}, \theta), \xi\big\rangle = b_0(\vec{y}, \theta, \xi), \quad \forall \xi \in W_0,$$

$$A_1 : c \in W_1 \to A_1(c) \in W_1' \quad \text{such that} \quad \big\langle A_1(c), \eta\big\rangle = a_0(c, \eta), \quad \forall \eta \in W_1,$$

$$B_1 : (\vec{y}, c) \in Y_0 \times W_1 \to B_1(\vec{y}, c) \in W_1' \quad \text{such that}$$
$$\quad \big\langle B_1(\vec{y}, c), \eta\big\rangle = b_0(\vec{y}, c, \eta), \quad \forall \eta \in W_1,$$

then, if $(\vec{y}, \theta, c)$ is a variational solution of (2), it satisfies

$$\begin{cases} \dfrac{d\vec{y}}{dt}(t) + \nu A\big(\vec{y}(t)\big) + B\big(\vec{y}(t)\big) = H(\theta)(t) \quad \text{in } Y_0', \text{ a.e. } t \in (0,T), \\ \dfrac{d\theta}{dt}(t) + k A_0\big(\theta(t)\big) + B_0\big(\vec{y}(t), \theta(t)\big) = G(t) + U(t) \quad \text{in } W_0', \text{ a.e. } t \in (0,T), \\ \dfrac{dc}{dt}(t) + \alpha A_1\big(c(t)\big) + B_1\big(\vec{y}(t), c(t)\big) = R(\theta, c)(t) \quad \text{in } W_1', \text{ a.e. } t \in (0,T), \\ \vec{y}(0) = \vec{\Phi}_0, \quad \theta(0) = \tau_0, \quad c(0) = \Psi_0, \end{cases} \tag{3}$$

where

$$\big\langle H(\theta)(t), \vec{z}\big\rangle_{Y_0', Y_0} = \big(\vec{f}(t) + \vec{\beta}(t)\theta(t), \vec{z}\big)_{L^2(\Omega)^3}, \quad \forall \vec{z} \in Y_0,$$

$$\big\langle G(t), \xi\big\rangle_{W_0', W_0} = \big(g(t), \xi\big)_{L^2(\Omega)}, \quad \forall \xi \in W_0,$$

$$\langle U(t),\xi\rangle_{W_0',W_0}=\big(u(t),\xi\big)_{L^2(\Gamma_1)},\quad \forall\xi\in W_0,$$

$$\langle R(\theta,c)(t),\eta\rangle_{W_1',W_1}=\big(r(\theta)(t)c(t),\eta\big)_{L^2(\Omega)},\quad \forall\eta\in W_1.$$

Lemma 4. *The variational formulation* (2) *admits, at least, a solution*

$$\begin{aligned}&(\vec{y},\theta,c)\in\big[L^2(0,T;Y_0)\cap L^\infty\big(0,T;L^2(\Omega)^3\big)\big]\\&\quad\times\big[L^2(0,T;W_0)\cap L^\infty\big(0,T;L^2(\Omega)\big)\big]\times\big[L^2(0,T;W_1)\cap L^\infty\big(0,T;L^2(\Omega)\big)\big]\end{aligned}$$

verifying the estimates

$$\begin{cases}\|\vec{y}\|_{L^2(0,T;Y_0)}+\|\vec{y}\|_{L^\infty(0,T;L^2(\Omega)^3)}\\ \quad\leqslant C_1\big(\|\vec{\Phi}_0\|_{L^2(\Omega)^3}+\|\vec{f}\|_{L^2(\Omega_T)^3}+\|\vec{\beta}\|_{L^\infty(\Omega_T)}\|\theta\|_{L^2(0,T;W_0)}\big),\\ \|\theta\|_{L^2(0,T;W_0)}+\|\theta\|_{L^\infty(0,T;L^2(\Omega))}\\ \quad\leqslant C_2\big(\|\tau_0\|_{L^2(\Omega)}+\|g\|_{L^2(\Omega_T)}+\|u\|_{L^2(\Sigma_T^1)}\big),\\ \|c\|_{L^2(0,T;V_0)}+\|c\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C_3\|\Psi_0\|_{L^2(\Omega)}.\end{cases}\tag{4}$$

Moreover,

$$\left(\frac{d\vec{y}}{dt},\frac{d\theta}{dt},\frac{dc}{dt}\right)\in L^1(0,T;Y_0')\times L^1(0,T;W_0')\times L^1(0,T;W_1')$$

verifies the estimates

$$\begin{cases}\left\|\dfrac{d\vec{y}}{dt}\right\|_{L^1(0,T;Y_0')}\\ \quad\leqslant C_4\big(\|\vec{y}\|^2_{L^2(0,T;Y_0)}+\|\vec{y}\|_{L^2(0,T;Y_0)}+\|\vec{\beta}\|^2_{L^\infty(\Omega_T)}+\|\theta\|^2_{L^2(0,T;W_0)}+\|\vec{f}\|_{L^2(\Omega_T)^3}\big),\\ \left\|\dfrac{d\theta}{dt}\right\|_{L^1(0,T;W_0')}\\ \quad\leqslant C_5\big(\|g\|_{L^2(\Omega_T)}+\|u\|_{L^2(\Gamma_T^1)}+\|\theta\|_{L^2(0,T;W_0)}+\|\theta\|^2_{L^2(0,T;W_0)}+\|\vec{y}\|^2_{L^2(0,T;Y_0)}\big),\\ \left\|\dfrac{dc}{dt}\right\|_{L^1(0,T;V_0')}\\ \quad\leqslant C_6\big(\|c\|_{L^2(0,T;V_0)}+\|c\|^2_{L^2(0,T;V_0)}+\|\vec{y}\|^2_{L^2(0,T;Y_0)}+\big\|r(\theta)\big\|^2_{L^\infty(\Omega_T)}\big).\end{cases}\tag{5}$$

Thus,

$$(\vec{y},\theta,c)\in\mathcal{C}_w\big([0,T];H\big)\times\mathcal{C}_w\big([0,T];L^2(\Omega)\big)\times\mathcal{C}_w\big([0,T];L^2(\Omega)\big).$$

Proof. Existence of solution can be obtained, by using the Galerkin method, in a similar way to Temam [16] or Ladyzhenskaya [13]. In the same way, this method allows us to obtain the following energy inequalities, by passing to the limit in the discrete problem:

$$\begin{cases} \|\vec{y}(t)\|^2_{L^2(\Omega)^3} + 2\nu \displaystyle\int_0^t a\big(\vec{y}(s), \vec{y}(s)\big)\, ds \\ \quad \leqslant \|\vec{\Phi}_0\|^2_{L^2(\Omega)^3} + 2 \displaystyle\int_0^t \big(\vec{f}(s) + \vec{\beta}(s)\theta(s), \vec{y}(s)\big)_{L^2(\Omega)^3}\, ds \quad \text{a.e. } t \in (0,T), \\ \|\theta(t)\|^2_{L^2(\Omega)} + 2k \displaystyle\int_0^t a_0\big(\theta(s), \theta(s)\big)\, ds \\ \quad \leqslant \|\tau_0\|^2_{L^2(\Omega)} + 2 \displaystyle\int_0^t \big\{\big(g(s), \theta(s)\big)_{L^2(\Omega)} + \big(u(s), \theta(s)\big)_{L^2(\Gamma_1)}\big\}\, ds \quad \text{a.e. } t \in (0,T), \\ \|c(t)\|^2_{L^2(\Omega)} + 2\alpha \displaystyle\int_0^t a_0\big(c(s), c(s)\big)\, ds \\ \quad \leqslant \|\Psi_0\|^2_{L^2(\Omega)} + 2 \displaystyle\int_0^t \big(r(\theta)(s)c(s), c(s)\big)_{L^2(\Omega)}\, ds \quad \text{a.e. } t \in (0,T), \end{cases} \tag{6}$$

from which we can easily obtain estimates (4).

As a consequence of (3) we have

$$\begin{aligned} &\frac{d\vec{y}}{dt}(t) = -\nu A\big(\vec{y}(t)\big) - B\big(\vec{y}(t)\big) + H(\theta)(t) \quad \text{in } Y_0',\ \text{a.e. } t \in (0,T), \\ &\frac{d\theta}{dt}(t) = -kA_0\big(\theta(t)\big) - B_0\big(\vec{y}(t), \theta(t)\big) + G(t) + U(t) \quad \text{in } W_0',\ \text{a.e. } t \in (0,T), \\ &\frac{dc}{dt}(t) = -\alpha A_1\big(c(t)\big) - B_1\big(\vec{y}(t), c(t)\big) + R(\theta, c)(t) \quad \text{in } W_1',\ \text{a.e. } t \in (0,T), \end{aligned}$$

from which we can deduce, thanks to the continuity of b and b_0, that

$$\frac{d\vec{y}}{dt} \in L^1(0,T;Y_0'), \qquad \frac{d\theta}{dt} \in L^1(0,T;W_0'), \qquad \frac{dc}{dt} \in L^1(0,T;W_1').$$

So, $\vec{y} \in \mathcal{C}([0,T];Y_0')$, $\theta \in \mathcal{C}([0,T];W_0')$, and $c \in \mathcal{C}([0,T];W_1')$. Thus, since $\vec{y} \in L^\infty(0,T;H)$ and the injection $H \hookrightarrow Y_0'$ is continuous, we deduce that $\vec{y} \in \mathcal{C}_w([0,T];H)$ (see, for instance, Temam [16]). We also obtain the analogous result for $\theta, c \in \mathcal{C}_w([0,T];L^2(\Omega))$.

Finally, estimates (5) for the norm of the derivatives are again a direct consequence of the continuity of b and b_0. □

Remark 5. Although the solution $(\vec{y}, \theta, c)$ obtained by the Galerkin method verifies the energy estimates (6), it is not true that any variational solution of the formulation (2) will verify them. That is the main motivation to define a new type of solution: we will say that $(\vec{y}, \theta, c)$ is a weak solution of (2) if it is a variational solution of (2) that verifies the energy inequalities (6). In particular, any weak solution of (2) will satisfy the estimates (4) and (5).

Remark 6. Once a weak solution $(\vec{y}, \theta, c)$ of (2) has been obtained, we can find by standard techniques a pressure $\pi \in \mathcal{D}'(\Omega_T)$ such that $(\vec{y}, \theta, c, \pi)$ is a solution of (1) verifying the partial differential equations in the sense of distributions, the boundary conditions in the sense of traces, and the initial conditions weakly in $L^2(\Omega)^3$ and $L^2(\Omega)$.

Nevertheless, since the uniqueness of weak solution of the Navier–Stokes equations is still an open problem in the three-dimensional case, we cannot expect uniqueness for the weak solution of our system (2). But, if we have a weak solution verifying the additional regularity $\vec{y} \in L^8(0, T; L^4(\Omega)^3)$ (a so-called strong solution), then this solution is unique (cf. Temam [16]). Unfortunately, the theory of existence and uniqueness of solutions is not complete: we do not know whether the weak solution is unique; we do not know whether a strong solution exists. However, in the next section we will prove an abstract result that will allow us to determine the uniqueness of the temperature component θ of the variational solution, in spite of the lack of regularity of the other two components $\vec{y}$ and c, and the fact that θ satisfies the energy estimate for the temperature, which will not be true for the other two components of the variational solution.

To avoid the troubles derived from the lack of uniqueness (which introduces a great difficulty in the obtaining of optimal solutions) the authors Alvarez-Vázquez and Martínez have considered in a previous paper [3] (also related to the food sterilization problem) the inclusion in the cost function of a term minimizing the turbulence (this assumption is suitable, for instance, for usual drinks like wine, beer, juices...) through the norm of the vorticity. Thanks to this term, it can be proved that each optimal solution is a strong solution and, consequently, is unique. This allows the authors to derive, in a direct way, the existence and uniqueness results.

However, in our analysis of the problem, we will not introduce a cost term for the vorticity and, consequently, we will have to deal with an optimal control problem whose state system has multiple solutions, that is, a so-called multistate control problem.

Recalling the original optimization problem, our main aim consists of minimizing the heat flux through the boundary Γ_1 (or, equivalently, minimizing the energy cost), in such a way that the mean microorganism concentration in a chosen zone A inside the can Ω be lower than a fixed threshold. This problem can be formulated as the following boundary control problem:

$$(P) \qquad \min_{(\vec{y}, \theta, c, u) \in \mathcal{U}} J(\vec{y}, \theta, c, u),$$

where the cost function $J(\vec{y}, \theta, c, u) = \frac{1}{2}\|u\|^2_{L^2(\Gamma^1_T)}$ is defined in the set

$$\mathcal{U} = \Bigg\{ (\vec{y}, \theta, c, u) \in L^2(0, T; Y_0) \times L^2(0, T; W_0) \times L^2(0, T; W_1) \times U_{ad} \\ \text{such that } (\vec{y}, \theta, c, u) \text{ is a weak solution of (2)}, \\ \text{and verifies } \frac{1}{\operatorname{meas}(A \times (0, T))} \int_{A \times (0,T)} c\, dx\, dt \leqslant \sigma \Bigg\},$$

for A an open subset of Ω, $U_{ad} \subset L^2(\Gamma_T^1)$ a non-empty, closed, convex set incorporating the control bound constraints, and $\sigma > 0$ a fixed threshold.

3. A partial uniqueness result for the state

In this section we will prove an abstract result of existence and uniqueness of solution for a parabolic equation with convective term, where the "lack of integrability" of the velocity does not allows us to demonstrate the uniqueness of solution by standard techniques. In the proof we will use similar arguments to those of Boccardo et al. [5] and Blanchard and Murat [4] for the case of renormalised solutions.

Theorem 7. *Let $\Omega \subset \mathbb{R}^3$ be a domain with boundary Γ smooth enough, and let us consider the following parabolic equation*:

$$\begin{cases} \dfrac{\partial \theta}{\partial t} - k\Delta\theta + \vec{w} \cdot \nabla\theta = g & \text{in } \Omega_T, \\ \theta = 0 & \text{on } \Gamma_T^0, \\ \dfrac{\partial \theta}{\partial n} = u & \text{on } \Gamma_T^1, \\ \theta(0) = \tau_0 & \text{in } \Omega, \end{cases} \tag{7}$$

where $g \in L^2(0,T; L^2(\Omega))$, $u \in L^2(\Gamma_T^1)$, and $\vec{w} \in L^2(0,T; L^3_\sigma(\Omega))$, with

$$L^3_\sigma(\Omega) = \overline{\{\vec{v} \in \mathcal{D}(\Omega)^3\colon\ \operatorname{div}(\vec{v}) = 0\}}^{L^3(\Omega)^3}.$$

Then, there exists a unique $\theta \in L^2(0,T; W_0) \cap L^\infty(0,T; L^2(\Omega))$ verifying the variational formulation of (7):

$$\begin{cases} \dfrac{d}{dt}\big(\theta(t), \xi\big)_{L^2(\Omega)} + ka_0\big(\theta(t), \xi\big) + b_0\big(\vec{w}(t), \theta(t), \xi\big) \\ \quad = \big(g(t), \xi\big)_{L^2(\Omega)} + \big(u(t), \xi\big)_{L^2(\Gamma_T^1)}, \quad \forall \xi \in W_0, \ \text{in } \mathcal{D}'(0,T), \\ \theta(0) = \tau_0, \end{cases} \tag{8}$$

and satisfying the following energy inequality:

$$\begin{aligned} &\|\theta(t)\|^2_{L^2(\Omega)} + 2k\int_0^t a_0\big(\theta(s), \theta(s)\big)\, ds \\ &\quad \leqslant \|\tau_0\|^2_{L^2(\Omega)} + 2\int_0^t \big\{\big(g(s), \theta(s)\big)_{L^2(\Omega)} + \big(u(s), \theta(s)\big)_{L^2(\Gamma_1)}\big\}\, ds \quad \text{a.e. } t \in (0,T). \end{aligned} \tag{9}$$

Moreover, $\theta \in \mathcal{C}([0,T]; L^p(\Omega))$, $\forall p \in [1,2)$.

Remark 8. If $\theta \in L^2(0,T;W_0)$, then $\vec{w}\cdot\nabla\theta \in L^1(0,T;L^{\frac{5}{6}}(\Omega))$ and, consequently, the following integral makes sense:

$$\int_0^T \int_\Omega \vec{w}(t)\cdot\nabla\theta(t)\xi\phi(t)\,dx\,dt, \quad \forall\xi \in W_0,\ \forall\phi \in \mathcal{D}(0,T),$$

due to the injection $W_0 \subset L^6(\Omega)$.

Moreover,

$$\int_\Omega \vec{v}\cdot\nabla\xi\xi\,dx = 0, \quad \forall\vec{v} \in L^3_\sigma(\Omega),\ \forall\xi \in W_0,$$

since the mapping $\vec{v} \in L^3_\sigma(\Omega) \to \int_\Omega \vec{v}\cdot\nabla\xi\xi\,dx \in \mathbb{R}$ is continuous for all $\xi \in W_0$.

Finally, we must remark that, if $\theta \in L^2(0,T;W_0)$ verifies the variational formulation (8), then

$$\frac{d\theta}{dt} \in L^2(0,T;W_0') + L^1\big(0,T;L^{\frac{5}{6}}(\Omega)\big).$$

Thus, we will not be able to obtain in the standard way *a priori* estimates for the solutions of (8), since we cannot use as test function the own solution. This will make harder to demonstrate the uniqueness of solution. In order to avoid this difficulty, we will construct, by the Galerkin method, a solution of (8) verifying (9) and, then, we will prove that this solution is unique.

In order to prove the above theorem we will need the following technical results:

Lemma 9. *Let us consider the following functional space*:

$$\mathcal{K} = \left\{\theta \in L^2(0,T;W_0)\colon\ \frac{d\theta}{dt} \in L^2(0,T;W_0') + L^1(\Omega_T)\right\}.$$

Then, $\mathcal{K} \subset \mathcal{C}([0,T];L^1(\Omega))$, *and*

$$\|\theta\|_{\mathcal{C}([0,T];L^1(\Omega))} \leqslant C\big(\|\theta\|_{L^2(0,T;W_0)} + \|\alpha\|_{L^2(0,T;W_0')} + \|\beta\|_{L^1(\Omega_T)}\big),$$

where $\frac{d\theta}{dt} = \alpha + \beta$, *with* $\alpha \in L^2(0,T;W_0')$ *and* $\beta \in L^1(\Omega_T)$.

Proof. The proof will be developed into three steps.

Step 1. As a first step we will construct a linear prolongation operator

$$P:\mathcal{K} \to \left\{z \in L^2(\mathbb{R};W_0)\colon\ \frac{dz}{dt} \in L^2(\mathbb{R};W_0') + L^1\big(\mathbb{R};L^1(\Omega)\big)\right\}$$

such that

$$P(\theta)|_{(0,T)} = \theta, \quad \forall \theta \in \mathcal{K},$$
$$\|P(\theta)\|_{L^2(\mathbb{R};W_0)} \leqslant C_1 \|\theta\|_{L^2(0,T;W_0)},$$
$$\left\|\frac{dP(\theta)}{dt}\right\|_{L^2(\mathbb{R};W_0')+L^1(\mathbb{R};L^1(\Omega))} \leqslant C_2\left(\|\theta\|_{L^2(0,T;W_0)} + \left\|\frac{d\theta}{dt}\right\|_{L^2(0,T;W_0')+L^1(\Omega_T)}\right).$$

In order to construct this operator we begin with a zero prolongation: We consider a function $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\eta(t) = \begin{cases} 1 & \text{if } t \leqslant \frac{T}{4}, \\ 0 & \text{if } t \geqslant \frac{3T}{4}, \end{cases}$$

and $0 \leqslant \eta(t) \leqslant 1$, $\forall t \in \mathbb{R}$. For an arbitrary mapping f defined in $(0,T)$ we define its zero prolongation $\tilde{f}$ defined in $(0,\infty)$ by

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t < T, \\ 0 & \text{if } t \geqslant T. \end{cases}$$

We have that $\eta\tilde{\theta} \in L^2(0,\infty;W_0)$ and $\frac{d(\eta\tilde{\theta})}{dt} = \eta\widetilde{\frac{d\theta}{dt}} + \eta'\tilde{\theta} \in \mathcal{D}'((0,\infty);W_0' + L^1(\Omega))$. On the other hand, since $\theta \in \mathcal{K}$, we have

$$\frac{d\theta}{dt} = \alpha + \beta, \quad \text{with } \alpha \in L^2(0,T;W_0') \text{ and } \beta \in L^1(\Omega_T).$$

So,

$$\widetilde{\frac{d\theta}{dt}} = \tilde{\alpha} + \tilde{\beta}, \quad \text{with } \tilde{\alpha} \in L^2(0,\infty;W_0') \text{ and } \tilde{\beta} \in L^1\big(0,\infty;L^1(\Omega)\big).$$

Thus,

$$\frac{d(\eta\tilde{\theta})}{dt} = (\eta'\tilde{\theta} + \eta\tilde{\alpha}) + \eta\tilde{\beta} \in L^2(0,\infty;W_0') + L^1\big(0,\infty;L^1(\Omega)\big),$$

from which we obtain the estimate for the norm of $\frac{d(\eta\tilde{\theta})}{dt}$ in the sum space $L^2(0,\infty;W_0') + L^1(0,\infty;L^1(\Omega))$:

$$\left\|\frac{d(\eta\tilde{\theta})}{dt}\right\|_{L^2(0,\infty;W_0')+L^1(0,\infty;L^1(\Omega))} \leqslant C\left(\|\theta\|_{L^2(0,T;W_0)} + \left\|\frac{d\theta}{dt}\right\|_{L^2(0,T;W_0)+L^1(\Omega_T)}\right).$$

Once the zero prolongation is defined we follow with a reflection: For any $u \in L^2(0,\infty;W_0)$ with $\frac{du}{dt} \in L^2(0,\infty;W_0') + L^1(0,\infty;L^1(\Omega))$ we define its prolongation by reflection by

$$u^*(t) = \begin{cases} u(t) & \text{if } t > 0, \\ u(-t) & \text{if } t < 0. \end{cases}$$

We have (see Brezis [6]) that

$$\frac{du^*}{dt} = \begin{cases} \frac{du}{dt}(t) & \text{if } t > 0, \\ -\frac{du}{dt}(-t) & \text{if } t < 0. \end{cases}$$

Moreover

$$\begin{aligned} \|u^*\|_{L^2(\mathbb{R};\, W_0)} &\leqslant \sqrt{2}\|u\|_{L^2(0,\infty;\, W_0)}, \\ \left\|\frac{du^*}{dt}\right\|_{L^2(\mathbb{R};\, W_0')+L^1(\mathbb{R};\, L^1(\Omega))} &\leqslant 2\left\|\frac{du}{dt}\right\|_{L^2(0,\infty;\, W_0)+L^1(0,\infty;\, L^1(\Omega))}. \end{aligned}$$

Then, if we denote $v_1 = (\eta\tilde{\theta})^* \in L^2(\mathbb{R}; W_0)$ and $v_2 = ((1-\eta)\hat{\theta})^{**}$ (where $\hat{\theta} \in L^2(-\infty, T; W_0)$ is the analogous zero prolongation for $t \leqslant 0$ of $\theta \in L^2(0, T; W_0)$ and $u^{**} \in L^2(\mathbb{R}; W_0)$ is the analogous reflection of $u \in L^2(-\infty, T; W_0)$ with respect to T), we will define $P(\theta) = v_1 + v_2$. Clearly, this prolongation operator verifies previous properties and $\operatorname{supp}(P(\theta)) \subset [-T, 2T]$.

Step 2. As a second step we will use a convolution approximation. We consider a sequence of functions $\{\rho_n\}_{n\in\mathbb{N}}$ such that

$$\begin{gathered} \rho_n \in \mathcal{C}_c^\infty(\mathbb{R}), \quad \forall n \in \mathbb{N}, \\ \operatorname{supp}(\rho_n) \subset B\left(0, \frac{1}{n}\right), \quad \forall n \in \mathbb{N}, \\ \rho_n \geqslant 0, \quad \forall n \in \mathbb{N}, \\ \int_{\mathbb{R}} \rho_n \, dx = 1, \quad \forall n \in \mathbb{N}. \end{gathered}$$

From classical functional results (see, for instance, [6]) we know that for all $n \in \mathbb{N}$:

$$\begin{aligned} \theta_n &= \rho_n * P(\theta) \in \mathcal{C}^\infty(\mathbb{R}; W_0), \\ \frac{d\theta_n}{dt} &= \rho_n * \frac{dP(\theta)}{dt} = \underbrace{\rho_n * \alpha}_{\alpha_n} + \underbrace{\rho_n * \beta}_{\beta_n} \in L^2(\mathbb{R}; W_0') + L^1\big(\mathbb{R}; L^1(\Omega)\big), \end{aligned}$$

where $\frac{dP(\theta)}{dt} = \alpha + \beta$, with $\alpha \in L^2(\mathbb{R}; W_0')$ and $\beta \in L^1(\mathbb{R}; L^1(\Omega))$. Moreover,

$$\begin{aligned} &\theta_n \to P(\theta) \quad \text{in } L^2(\mathbb{R}; W_0), \\ &\frac{d\theta_n}{dt} \to \frac{dP(\theta)}{dt} \quad \text{in } L^2(\mathbb{R}; W_0') + L^1\big(\mathbb{R}; L^1(\Omega)\big), \\ &\alpha_n \to \alpha \quad \text{in } L^2(\mathbb{R}; W_0'), \\ &\beta_n \to \beta \quad \text{in } L^1\big(\mathbb{R}; L^1(\Omega)\big), \end{aligned}$$

and $\operatorname{supp}(\theta_n) \subset [-2T, 3T]$ for n large enough.

Step 3. As a final step we will obtain the boundedness in $\mathcal{C}([0,T];L^1(\Omega))$. For each $\delta>0$, we define the following mapping:

$$\gamma_\delta(r)=\begin{cases}1 & \text{if } r\geqslant\delta,\\ \frac{r}{\delta} & \text{if } -\delta\leqslant r\leqslant\delta,\\ -1 & \text{if } r\leqslant-\delta.\end{cases}$$

Its primitive is given by the expression

$$K_\delta(r)=\int_0^r\gamma_\delta(s)\,ds=\begin{cases}r-\frac{\delta}{2} & \text{if } r\geqslant\delta,\\ \frac{r^2}{2\delta} & \text{if } -\delta\leqslant r\leqslant\delta,\\ -r-\frac{\delta}{2} & \text{if } r\leqslant-\delta.\end{cases}$$

A simple computation also gives that $0\leqslant|r|-K_\delta(r)\leqslant\frac{\delta}{2}$, $\forall r\in\mathbb{R}$.

Let us see now that $\gamma_\delta(\theta_n)\in L^2(\mathbb{R};W_0)$. To do that, we will first prove that for all $u\in W_0$, $\gamma_\delta(u)\in W_0$. Since

$$W_0=\overline{\{v\in\mathcal{D}(\overline{\Omega}):\ v|_{\Gamma_0}=0\}}^{H^1(\Omega)},$$

for all $u\in W_0$ there will be a sequence $\{u_n\}_{n\in\mathbb{N}}\subset\{v\in\mathcal{D}(\overline{\Omega}):\ v|_{\Gamma_0}=0\}$ such that $u_n\to u$ in $H^1(\Omega)$. Moreover, $\nabla\gamma_\delta(u_n)=\gamma'_\delta(u_n)\nabla u_n$, from which $\gamma_\delta(u_n)\in H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})$ with $\gamma_\delta(u_n)|_{\Gamma_0}=0$. Thus, $\gamma_\delta(u_n)\in W_0$. On the other hand,

$$\left.\begin{array}{l}\gamma_\delta(u_n)\to\gamma_\delta(u)\quad\text{in } L^2(\Omega),\\ \left\|\nabla\gamma_\delta(u_n)\right\|_{L^2(\Omega)^3}\leqslant\dfrac{1}{\delta}\|u_n\|_{L^2(\Omega)^3}\end{array}\right\}\quad\Rightarrow\quad\gamma_\delta(u_n)\rightharpoonup\gamma_\delta(u)\quad\text{in } H^1(\Omega),$$

and, since W_0 is weakly closed in $H^1(\Omega)$, we obtain that $\gamma_\delta(u)\in W_0$. Finally, from the estimate $\|\gamma_\delta(u)\|_{W_0}\leqslant C\|u\|_{W_0}$, we can obtain that the function $\gamma_\delta:L^2(\mathbb{R};W_0)\to L^2(\mathbb{R};W_0)$ is well defined and demicontinuous (continuous from the strong topology to the weak one).

Now, from the results of Kavian [11], we have that the mapping $Q_\delta:v\in L^2(\Omega)\to Q_\delta(v)=\int_\Omega K_\delta(v(x))\,dx\in\mathbb{R}$ is differentiable with $\langle Q'_\delta(u),v\rangle=\int_\Omega\gamma_\delta(u)v\,dx$. Then, from the fact that $\theta_n\in\mathcal{C}^\infty(\mathbb{R};W_0)$, we can say that

$$\frac{d}{dt}\int_\Omega K_\delta\bigl(\theta_n(t)\bigr)\,dx=\int_\Omega\gamma_\delta\bigl(\theta_n(t)\bigr)\frac{d\theta_n}{dt}(t)\,dx=\left\langle\frac{d\theta_n}{dt}(t),\gamma_\delta(\theta_n)(t)\right\rangle_{W'_0,W_0}.$$

So, integrating in $[-2T,t]$ for $\theta_n-\theta_m$:

$$\begin{aligned}\int_\Omega K_\delta\bigl(\theta_n(t)-\theta_m(t)\bigr)\,dx&=\int_{-2T}^t\bigl\langle\alpha_n(s)-\alpha_m(s),\gamma_\delta\bigl(\theta_n(s)-\theta_m(s)\bigr)\bigr\rangle_{W_0,W'_0}\,ds\\&\quad+\int_{-2T}^t\int_\Omega\bigl(\beta_n(s)-\beta_m(s)\bigr)\gamma_\delta\bigl(\theta_n(s)-\theta_m(s)\bigr)\,dx\,ds\end{aligned}$$

$$\begin{aligned}&\leqslant \|\alpha_n-\alpha_m\|_{L^2(-2T,3T;W_0')}\big\|\gamma_\delta(\theta_n-\theta_m)\big\|_{L^2(-2T,3T;W_0)}\\&\quad+\|\beta_n-\beta_m\|_{L^1(-2T,3T;L^1(\Omega))}\big\|\gamma_\delta(\theta_n-\theta_m)\big\|_{L^\infty(\Omega_T)}.\end{aligned}\tag{10}$$

Now, since

$$\nabla\gamma_\delta(\theta_n-\theta_m)=\begin{cases}\gamma_\delta'(\theta_n-\theta_m)\nabla(\theta_n-\theta_m) & \text{if } \theta_n-\theta_m\notin\{-\delta,\delta\},\\ 0 & \text{if } \theta_n-\theta_m\in\{-\delta,\delta\},\end{cases}$$

we obtain that

$$\big\|\gamma_\delta(\theta_n-\theta_m)\big\|_{L^2(-2T,3T;W_0)}\leqslant\frac{C}{\delta}\|\theta_n-\theta_m\|_{L^2(-2T,3T;W_0)}.$$

Taking the supremum in the above expression (10):

$$\begin{aligned}0&\leqslant\sup_{t\in[-2T,3T]}\int_\Omega K_\delta(\theta_n-\theta_m)\,dx\\&\leqslant\frac{C}{\delta}\|\theta_n-\theta_m\|_{L^2(\mathbb{R};W_0)}\|\alpha_n-\alpha_m\|_{L^2(\mathbb{R};W_0')}+\|\beta_n-\beta_m\|_{L^1(\mathbb{R};L^1(\Omega))},\end{aligned}$$

but, $|r|\leqslant K_\delta(r)+\frac{\delta}{2}$, from which:

$$\begin{aligned}\sup_{t\in[0,T]}\int_\Omega\big|\theta_n(x,t)-\theta_m(x,t)\big|\,dx&\leqslant\frac{C}{\delta}\|\alpha_n-\alpha_m\|_{L^2(\mathbb{R};W_0')}\|\theta_n-\theta_m\|_{L^2(\mathbb{R};W_0)}\\&\quad+\frac{\delta}{2}\operatorname{meas}(\Omega)+\|\beta_n-\beta_m\|_{L^1(\mathbb{R};L^1(\Omega))},\quad\forall\delta>0.\end{aligned}$$

Then, taking $\delta=\sqrt{\frac{2C\|\theta_n-\theta_m\|_{L^2(\mathbb{R};W_0)}\|\alpha_n-\alpha_m\|_{L^2(\mathbb{R};W_0')}}{\operatorname{meas}(\Omega)}}$, we have that

$$\begin{aligned}&\|\theta_n-\theta_m\|_{\mathcal{C}([0,T];L^1(\Omega))}\\&\quad\leqslant C\big(\|\theta_n-\theta_m\|_{L^2(\mathbb{R};W_0)}+\|\alpha_n-\alpha_m\|_{L^2(\mathbb{R};W_0')}+\|\beta_n-\beta_m\|_{L^1(\mathbb{R};L^1(\Omega))}\big).\end{aligned}$$

This fact, together with the convergence of sequences $\{\theta_n\}_{n\in\mathbb{N}}$, $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R};W_0)$, $L^2(\mathbb{R};W_0')$ and $L^1(\mathbb{R};L^1(\Omega))$, respectively, assures that $\{\theta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1(\Omega))$ and, consequently, $\theta\in\mathcal{C}([0,T];\ L^1(\Omega))$. Finally, analogous computations allow us to obtain the estimate

$$\|\theta\|_{\mathcal{C}([0,T];L^1(\Omega))}\leqslant C\big(\|\theta\|_{L^2(0,T;W_0)}+\|\alpha\|_{L^2(0,T;W_0')}+\|\beta\|_{L^1(\Omega_T)}\big).\qquad\square$$

Corollary 10. *If $\theta\in\mathcal{K}\cap L^\infty(0,T;L^2(\Omega))$, then $\theta\in\mathcal{C}([0,T];L^p(\Omega))$, $\forall p\in[1,2)$.*

Proof. Let us take the same sequence $\{\theta_n\}_{n\in\mathbb{N}}$ from the proof of previous lemma, and let us consider the following interpolation inequality:

$$\left\|\theta_n(t)-\theta_m(t)\right\|_{L^p(\Omega)} \leqslant C\left\|\theta_n(t)-\theta_m(t)\right\|_{L^1(\Omega)}^{\vartheta}\left\|\theta_n(t)-\theta_m(t)\right\|_{L^2(\Omega)}^{1-\vartheta},$$

valid for all $1 \leqslant p < 2$, and $\frac{1}{p} = \vartheta + \frac{1-\vartheta}{2}$, with $\vartheta \in [0,1]$. Then, since $\theta \in L^\infty(0,T;L^2(\Omega))$, we have

$$\sup_{t\in\mathbb{R}}\left\|\theta_n(t)\right\|_{L^2(\Omega)} \leqslant \|\theta\|_{L^\infty(0,T;L^2(\Omega))},$$

which allows us to deduce:

$$\sup_{t\in[0,T]}\left\|\theta_n(t)-\theta_m(t)\right\|_{L^p(\Omega)} \leqslant C\sup_{t\in[0,T]}\left\|\theta_n(t)-\theta_m(t)\right\|_{L^1(\Omega)}^{\vartheta}.$$

Thus, $\{\theta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T];L^p(\Omega))$, $\forall p \in [1,2)$, and, consequently, $\theta \in \mathcal{C}([0,T];L^p(\Omega))$, $\forall p \in [1,2)$. □

Corollary 11. *If $\theta \in \mathcal{K}$, then*

$$\int_\Omega K_\delta\bigl(\theta(t)\bigr)\,dx - \int_\Omega K_\delta\bigl(\theta(0)\bigr)\,dx$$
$$= \int_0^t\int_\Omega \beta(s)\gamma_\delta\bigl(\theta(s)\bigr)\,dx\,ds + \int_0^t \bigl\langle \alpha(s), \gamma_\delta\bigl(\theta(s)\bigr)\bigr\rangle\,ds, \quad \forall t \in [0,T].$$

Proof. It is enough to consider the corresponding formula for the elements of the sequence $\{\theta_n\}_{n\in\mathbb{N}}$ and pass to the limit, using its convergence in $\mathcal{C}([0,T];L^1(\Omega)) \cap L^2(0,T;W_0)$, the continuity (see [11]) of the mapping Q_δ, and the following convergence:

$$\begin{aligned}
\beta_n \to \beta \quad &\text{in } L^1\bigl(0,T;L^1(\Omega)\bigr),\\
\alpha_n \to \alpha \quad &\text{in } L^2(0,T;W_0'),\\
\gamma_\delta(\theta_n) \rightharpoonup \gamma_\delta(\theta) \quad &\text{in } L^2(0,T;W_0),
\end{aligned}$$

where the last convergence is a direct consequence of the demicontinuity of the mapping γ_δ. □

After these technical results we are now able to prove the main Theorem 7.

Proof of Theorem 7. For the existence we will use a classical Galerkin approach. Let $\{w_1,\ldots,w_m,\ldots\}$ be a basis of W_0. We denote $W_m = \langle\{w_1,\ldots,w_m\}\rangle$ and consider a sequence $\{\tau_{0,m}\}_{m\in\mathbb{N}}$ such that $\tau_{0,m} \in W_m$ and $\tau_{0,m} \to \tau_0$ in $L^2(\Omega)$.

We consider then the approximated problem corresponding to (8), consisting of finding $\theta_m(t) = \sum_{j=1}^m g_{i,m}(t) w_i \in W_m$ solution of

$$\begin{cases} \dfrac{d}{dt}\big(\theta_m(t), w_i\big)_{L^2(\Omega)} + ka_0\big(\theta_m(t), w_i\big) + b_0\big(\vec{w}(t), \theta_m(t), w_i\big) \\ \quad = \big(g(t), w_i\big)_{L^2(\Omega)} + \big(u(t), w_i\big)_{L^2(\Gamma_1)}, \quad \forall i = 1, \ldots, m, \\ \theta_m(0) = \tau_{0,m}. \end{cases} \tag{11}$$

If we introduce the following notations:

$$\begin{aligned} \big(C_m^0\big)_{i,j} &= (w_j, w_i)_{L^2(\Omega)}, \quad \forall i, j = 1, \ldots, m, \\ \big(B_m^0(t)\big)_{i,j} &= b_0\big(\vec{w}(t), w_j, w_i\big), \quad \forall i, j = 1, \ldots, m, \\ \big(A_m^0\big)_{i,j} &= ka_0(w_j, w_i), \quad \forall i, j = 1, \ldots, m, \\ \big(\vec{g}_m(t)\big)_i &= g_{i,m}(t), \quad \forall i = 1, \ldots, m, \\ \big(\vec{d}_m^0(t)\big)_i &= \big(g(t), w_i\big)_{L^2(\Omega)} + \big(u(t), w_i\big)_{L^2(\Gamma_1)}, \quad \forall i = 1, \ldots, m, \end{aligned}$$

$$\vec{\tau}_{0,m} \quad \text{such that} \quad \tau_{0,m} = \sum_{i=1}^m (\vec{\tau}_{0,m})_i w_i,$$

we have that (11) is equivalent to:

$$\begin{cases} \dfrac{d\vec{g}_m}{dt}(t) = -\big(C_m^0\big)^{-1}\big(A_m^0 + B_m^0(t)\big)\vec{g}_m(t) + \big(C_m^0\big)^{-1}\vec{d}_m^0(t) \quad \text{a.e. } t \in [0, T], \\ \vec{g}_m(0) = \vec{\tau}_{0,m}. \end{cases} \tag{12}$$

In order to demonstrate that problem (12) has a unique solution $\vec{g}_m : [0, T] \to \mathbb{R}^m$, we only need to prove that the mapping $T : \vec{\eta} \in \mathcal{C}([0, T]; \mathbb{R}^m) \to T(\vec{\eta}) \in \mathcal{C}([0, T]; \mathbb{R}^m)$ given by

$$T(\vec{\eta})(t) = \vec{\tau}_{0,m} - \int_0^t \big(C_m^0\big)^{-1}\big(A_m^0 + B_m^0(s)\big)\vec{\eta}(s)\, ds + \int_0^t \big(C_m^0\big)^{-1}\vec{d}_m^0(s)\, ds$$

has a fixed point. In $\mathcal{C}([0, T]; \mathbb{R}^m)$, we will consider the following norm:

$$\|\vec{\eta}\|_B = \sup_{0 \leqslant t \leqslant T} \big\{ e^{-kt} \big\|\vec{\eta}(t)\big\|_{\mathbb{R}^m} \big\},$$

with a suitable $k > 0$. Let $A_m^*(t) = (C_m^0)^{-1}(A_m^0 + B_m^0(t))$. Since $\vec{w} \in L^2(0, T; L^3_\sigma(\Omega))$, we have that $A_m^* \in L^2(0, T; M_{m\times m}(\mathbb{R}))$, and

$$\left\| \int_0^t \big(C_m^0\big)^{-1}\big(A_m^0 + B_m^0(s)\big)\vec{\eta}(s)\, ds \right\|_{\mathbb{R}^m} \leqslant \frac{e^{kt}}{\sqrt{2k}} \big\|A_m^*\big\|_{L^2(0,T;M_{m\times m}(\mathbb{R}))} \|\vec{\eta}\|_B.$$

Thus,

$$\sup_{0\leqslant t\leqslant T}\left\{e^{-kt}\left\|\int_0^t \left(C_m^0\right)^{-1}\left(A_m^0+B_m^0(s)\right)\vec{\eta}(s)\,ds\right\|_{\mathbb{R}^m}\right\}\leqslant\frac{1}{\sqrt{2k}}\|A_m^*\|_{L^2(0,T;M_{m\times m}(\mathbb{R}))}\|\vec{\eta}\|_B,$$

from which, taking k large enough such that $\frac{\|A_m^*\|_{L^2(0,T;M_{m\times m}(\mathbb{R}))}}{\sqrt{2k}}<1$, we obtain that mapping T is contractive in space $\mathcal{C}([0,T];\mathbb{R}^m)$ endowed with the norm $\|\cdot\|_B$, and, consequently, it has a unique fixed point $\vec{g}_m\in\mathcal{C}([0,T];\mathbb{R}^m)$, solution of (12).

Now, multiplying (11) by $(\vec{g}_m)_i$ y summing in i, we obtain that

$$\frac{d}{dt}\left(\theta_m(t),\theta_m(t)\right)_{L^2(\Omega)}+ka_0\left(\theta_m(t),\theta_m(t)\right)=\left(g(t),\theta_m(t)\right)_{L^2(\Omega)}+\left(u(t),\theta_m\right)_{L^2(\Gamma_1)}. \tag{13}$$

Integrating in $[0,t]$, we achieve the following energy equality for θ_m:

$$\begin{aligned}&\left\|\theta_m(t)\right\|^2_{L^2(\Omega)}+2k\int_0^t\left\|\theta_m(s)\right\|^2_{W_0}ds\\&\quad=\|\tau_{0,m}\|^2+2\int_0^t\left\langle g(s),\tau_m(s)\right\rangle_{L^2(\Omega)}ds+2\int_0^t\int_{\Gamma_1}u(x,s)\theta_m(x,s)\,dx\,ds,\end{aligned} \tag{14}$$

from which we easily obtain that

$$\frac{d}{dt}\left\|\theta_m(t)\right\|^2_{L^2(\Omega)}+\left\|\theta_m(t)\right\|^2_{W_0}\leqslant C\left(\left\|g(t)\right\|^2_{L^2(\Omega)}+\left\|u(t)\right\|^2_{L^2(\Gamma_T^1)}+\|\tau_{0,m}\|^2_{L^2(\Omega)}\right). \tag{15}$$

Thanks to (15) and the convergence of $\tau_{0,m}$ to τ_0 in $L^2(\Omega)$, we know that $\{\theta_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(0,T;W_0)\cap L^\infty(0,T;L^2(\Omega))$, thus, there exist a subsequence of $\{\theta_m\}_{m\in\mathbb{N}}$, still denoted in the same way, such that

$$\begin{aligned}&\theta_m\rightharpoonup\theta\quad\text{in } L^2(0,T;W_0),\\&\theta_m\rightharpoonup^*\theta\quad\text{in } L^\infty\left(0,T;L^2(\Omega)\right).\end{aligned}$$

These convergences allows us to pass to the limit in Eq. (11), obtaining that $\theta\in L^2(0,T;W_0)\cap\mathcal{C}_w([0,T];L^2(\Omega))$ is solution of (8).

Multiplying equality (14) by $\phi(t)$, with $\phi\in\mathcal{D}(0,T)$, $\phi\geqslant 0$, and integrating in $[0,T]$ we have

$$\begin{aligned}&\int_0^T\left\{\left\|\theta_m(t)\right\|^2_{L^2(\Omega)}+2k\int_0^t\left\|\theta_m(s)\right\|^2_{W_0}ds\right\}\phi(t)\,dt\\&\quad=\int_0^T\left\{\|\tau_{0,m}\|^2+2\int_0^t\left\langle g(s),\theta_m(s)\right\rangle_{L^2(\Omega)}ds+2\int_0^t\int_{\Gamma_1}u(x,s)\theta_m(x,s)\,dx\,ds\right\}\phi(t)\,dt,\end{aligned}$$

from which, passing to the limit, we can obtain the energy inequality (9).

Finally, in order to prove the uniqueness of the solution, it is enough to demonstrate that the unique solution of the homogeneous problem:

$$\begin{cases} \dfrac{d}{dt}\big(\theta(t),\xi\big)_{L^2(\Omega)} + ka_0\big(\theta(t),\xi\big) + b_0\big(\vec{w}(t),\theta(t),\xi\big) = 0, \quad \forall \xi \in W_0, \text{ a.e. } t \in (0,T), \\ \theta(0) = 0, \end{cases}$$

is the null function $\theta(x,t) = 0$ a.e. $(x,t) \in \Omega_T$. In order to prove this, we take as test function $\xi = \gamma_\delta(\theta(t))$. Then, by previous results, we have

$$\int_\Omega K_\delta\big(\theta(t)\big)\,dx + k\int_0^t\int_\Omega \nabla\theta(s)\cdot\nabla\big(\gamma_\delta\big(\theta(s)\big)\big)\,dx\,ds + \int_0^t\int_\Omega \vec{w}(s)\cdot\nabla\theta(s)\gamma_\delta\big(\theta(s)\big)\,dx\,ds = 0, \quad \text{a.e. } t \in (0,T).$$

Taking into account that

$$\int_\Omega \nabla\theta(t)\cdot\nabla\big(\gamma_\delta\big(\theta(t)\big)\big)\,dx = \int_\Omega \gamma_\delta'\big(\theta(t)\big)\big|\nabla\theta(t)\big|^2\,dx \geqslant 0, \quad \text{a.e. } t \in (0,T),$$

$$\int_\Omega \big(\vec{w}\cdot\nabla\theta(t)\big)\gamma_\delta\big(\theta(t)\big) = \int_\Omega \vec{w}(t)\cdot\nabla\big(K_\delta\big(\theta(t)\big)\big)\,dx = 0 \quad \text{a.e. } t \in (0,T),$$

we deduce that $\int_\Omega K_\delta(\theta(t))\,dx = 0$, a.e. $t \in (0,T)$, and, consequently, $\theta(x,t) = 0$, a.e. $(x,t) \in \Omega_T$. □

Remark 12. Summarizing, we have proved that the unique solution of the heat equation (8) is the one constructed by the Galerkin method, and that it verifies the energy estimate (9), in spite of the low regularity of the velocity $\vec{w} \in L^2(0,T;L^3_\sigma(\Omega))$.

Returning to our original state system (1), and as a direct consequence of the abstract Theorem 7, we can enunciate the following result of partial uniqueness for the state:

Theorem 13. *The temperature component $\theta \in L^2(0,T;W_0) \cap L^\infty(0,T;L^2(\Omega))$ of the solution of the variational formulation* (2) *is unique, and it satisfies the additional regularity $\theta \in \mathcal{C}([0,T];L^p(\Omega))$, $\forall p \in [1,2)$.*

4. Existence of optimal solutions

In order to obtain the existence of optimal solutions for problem (P) we will need this compactness result whose proof can be found in Temam [16]:

Lemma 14. *Let X_0, X and X_1 be three Hilbert spaces such that $X_0 \subset X \subset X_1$ with continuous injection, and $X_0 \subset X$ with compact injection. We consider the following space*:

$$\mathcal{W}(0,T;2,1;X_0,X_1)=\left\{v\in L^2(0,T;X_0),\ \frac{dv}{dt}\in L^1(0,T;X_1)\right\}$$

endowed with the norm

$$\|v\|_{\mathcal{W}(0,T;2,1;X_0,X_1)}=\|v\|_{L^2(0,T;X_0)}+\left\|\frac{dv}{dt}\right\|_{L^1(0,T;X_1)}.$$

Then, the injection $\mathcal{W}(0,T;2,1;X_0,X_1)\hookrightarrow L^2(0,T;X)$ is also compact.

Remark 15. The above result can be easily generalized for X_0 and X Banach spaces, so that the following injection is compact:

$$\mathcal{W}(0,T;\alpha_0,1;X_0,X_1)\hookrightarrow L^{\alpha_0}(0,T;X),$$

for any $\alpha_0\in(1,\infty)$.

As a consequence of the above lemma we have the following straightforward results:

Corollary 16. *The following injections are compact*:

$$\begin{aligned}\mathcal{W}(0,T;2,1;Y_0,Y_0')&\hookrightarrow L^2\big(0,T;L^p(\Omega)^3\cap H\big),\\ \mathcal{W}(0,T;2,1;W_i,W_i')&\hookrightarrow L^2\big(0,T;L^p(\Omega)\big),\quad i=0,1,\end{aligned}$$

for all $p\in[2,6)$.

Corollary 17. *Let $\{(\vec{y}_k,\theta_k,c_k,u_k)\}_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{W}(0,T;2,1;Y_0,Y_0')\times\mathcal{W}(0,T;2,1;W_0,W_0')\times\mathcal{W}(0,T;2,1;W_1,W_1')\times L^2(\Gamma_T^1)$ such that the following energy inequalities are satisfied*:

$$\begin{cases}\big\|\vec{y}_k(t)\big\|^2_{L^2(\Omega)^3}+2\nu\displaystyle\int_0^t a\big(\vec{y}_k(s),\vec{y}_k(s)\big)\,ds\\ \qquad\leqslant\|\vec{\Phi}_0\|^2_{L^2(\Omega)^3}+2\displaystyle\int_0^t\big(\vec{f}(s)+\vec{\beta}(s)\theta_k(s),\vec{y}_k(s)\big)_{L^2(\Omega)^3}\,ds\quad a.e.\ t\in(0,T),\\ \big\|c_k(t)\big\|^2_{L^2(\Omega)}+2\alpha\displaystyle\int_0^t a_0\big(c_k(s),c_k(s)\big)\,ds\\ \qquad\leqslant\|\Psi_0\|^2_{L^2(\Omega)}+2\displaystyle\int_0^t\big(r(\theta_k)(s)c_k(s),c_k(s)\big)_{L^2(\Omega)}\,ds\quad a.e.\ t\in(0,T),\end{cases}$$

for all $k\in\mathbb{N}$.

Then, there exist elements $(\vec{y},\theta,c,u)\in L^2(0,T;Y_0)\times L^2(0,T;W_0)\times L^2(0,T;W_1)\times L^2(\Gamma_T^1)$ such that

$$\begin{aligned}
&\vec{y}_k \rightharpoonup \vec{y} \quad \text{in } L^2(0,T;Y_0),\\
&\vec{y}_k \to \vec{y} \quad \text{in } L^2\big(0,T;L^p(\Omega)^3\big),\\
&\theta_k \rightharpoonup \theta \quad \text{in } L^2(0,T;W_0),\\
&\theta_k \to \theta \quad \text{in } L^2\big(0,T;L^p(\Omega)\big),\\
&c_k \rightharpoonup c \quad \text{in } L^2(0,T;W_1),\\
&c_k \to c \quad \text{in } L^2\big(0,T;L^p(\Omega)\big),
\end{aligned}$$

for all $p \in [2,6)$, *which allows us to pass to the limit in the previous estimate, obtaining that*

$$\begin{cases}
\big\|\vec{y}(t)\big\|^2_{L^2(\Omega)^3} + 2\nu \displaystyle\int_0^t a\big(\vec{y}(s),\vec{y}(s)\big)\,ds\\
\quad \leqslant \|\vec{\Phi}_0\|^2_{L^2(\Omega)^3} + 2\displaystyle\int_0^t \big(\vec{f}(s)+\vec{\beta}(s)\theta(s),\vec{y}(s)\big)_{L^2(\Omega)^3}\,ds \quad \text{a.e. } t\in(0,T),\\
\big\|c(t)\big\|^2_{L^2(\Omega)} + 2\alpha \displaystyle\int_0^t a_0\big(c(s),c(s)\big)\,ds\\
\quad \leqslant \|\Psi_0\|^2_{L^2(\Omega)} + 2\displaystyle\int_0^t \big(r(\theta)(s)c(s),c(s)\big)_{L^2(\Omega)}\,ds \quad \text{a.e. } t\in(0,T).
\end{cases}$$

Corollary 18. *Let* $\{(\vec{y}_k,\theta_k,c_k)\}$ *be a bounded sequence in* $\mathcal{W}(0,T;2,1;Y_0,Y_0') \times \mathcal{W}(0,T;2,1;W_0,W_0') \times \mathcal{W}(0,T;2,1;W_1,W_1')$. *Then, there exist elements* $(\vec{y},\theta,c) \in L^2(0,T;Y_0) \times L^2(0,T;W_0) \times L^2(0,T;W_1)$ *such that*

$$\begin{aligned}
&\vec{y}_k \rightharpoonup \vec{y} \quad \text{in } L^2(0,T;Y_0),\\
&\vec{y}_k \to \vec{y} \quad \text{in } L^2\big(0,T;L^4(\Omega)^3\big),\\
&\theta_k \rightharpoonup \theta \quad \text{in } L^2(0,T;W_0),\\
&c_k \rightharpoonup c \quad \text{in } L^2(0,T;W_1),
\end{aligned}$$

which allows us to obtain the following convergences:

$$\begin{aligned}
&\int_0^T b\big(\vec{y}_k(t),\vec{\theta}_k(t),\vec{\Psi}\otimes\phi(t)\big)\,dt \to \int_0^T b\big(\vec{y}(t),\vec{\theta}(t),\vec{\Psi}\otimes\phi(t)\big)\,dt,\\
&\int_0^T b_0\big(\vec{y}_k(t),\theta_k(t),\xi\otimes\phi(t)\big)\,dt \to \int_0^T b_0\big(\vec{y}(t),\theta(t),\xi\otimes\phi(t)\big)\,dt,
\end{aligned}$$

$$\int_0^T b_0\big(\vec{y}_k(t), c_k(t), \eta \otimes \phi(t)\big)\,dt \to \int_0^T b_0\big(\vec{y}(t), c(t), \eta \otimes \phi(t)\big)\,dt,$$

$\forall \vec{\Psi} \in Y_0$, $\forall \xi \in W_0$, $\forall \eta \in W_1$, and $\forall \phi \in \mathcal{D}(0,T)$.

Then, we can prove the following property for the set $\mathcal{U}$:

Lemma 19. *The set $\mathcal{U}$ is weakly closed.*

Proof. Let us consider a sequence $\{(\vec{y}_k, \theta_k, c_k, u_k)\}_{k\in\mathbb{N}} \subset \mathcal{U}$ such that

$$(\vec{y}_k, \theta_k, c_k, u_k) \rightharpoonup (\vec{y}, \theta, c, u) \quad \text{in } L^2(0,T;Y_0) \times L^2(0,T;W_0) \times L^2(0,T;W_1) \times L^2\big(\Gamma_T^1\big).$$

Thus, $\{(\vec{y}_k, \theta_k, c_k, u_k)\}_{k\in\mathbb{N}}$ will be bounded in $L^2(0,T;Y_0) \times L^2(0,T;W_0) \times L^2(0,T;W_1) \times L^2(\Gamma_T^1)$ and then, thanks to (4) and (5), it will be also bounded in $[\mathcal{W}(0,T;2,1;Y_0,Y_0') \cap L^\infty(0,T;L^2(\Omega)^3)] \times [\mathcal{W}(0,T;2,1;W_0,W_0') \cap L^\infty(0,T;L^2(\Omega))] \times [\mathcal{W}(0,T;2,1;W_1,W_1') \cap L^\infty(0,T;L^2(\Omega))] \times L^2(\Gamma_T^1)$.

On the other hand, $\{(\vec{y}_k, \theta_k, c_k, u_k)\}_{k\in\mathbb{N}}$ is a sequence of weak solutions of (2), so it must verify the state system for all $k \in \mathbb{N}$:

$$\begin{cases} \dfrac{d}{dt}\big(\vec{y}_k(t), \vec{\Psi}\big)_{L^2(\Omega)^3} + \nu a\big(\vec{y}_k(t), \vec{\Psi}\big) + b\big(\vec{y}_k(t), \vec{y}_k(t), \vec{\Psi}\big) \\ \quad = \big(\vec{f}(t) + \vec{\beta}(t)\theta_k(t), \vec{\Psi}\big)_{L^2(\Omega)^3}, \quad \forall \vec{\Psi} \in Y_0, \text{ in } \mathcal{D}'(0,T), \\ \dfrac{d}{dt}\big(\theta_k(t), \xi\big)_{L^2(\Omega)} + ka_0\big(\theta_k(t), \xi\big) + b_0\big(\vec{y}_k(t), \theta_k(t), \xi\big) \\ \quad = \big(g(t), \xi\big)_{L^2(\Omega)} + \big(u(t), \xi\big)_{L^2(\Gamma_T^1)}, \quad \forall \xi \in W_0, \text{ in } \mathcal{D}'(0,T), \\ \dfrac{d}{dt}\big(c_k(t), \eta\big)_{L^2(\Omega)} + b_0\big(\vec{y}_k(t), c_k(t), \eta\big) + \alpha a_0\big(c_k(t), \eta\big) + \big(r(\theta_k)(t)c_k(t), \eta\big)_{L^2(\Omega)} \\ \quad = 0, \quad \forall \eta \in W_1, \text{ in } \mathcal{D}'(0,T), \\ \vec{y}_k(0) = \vec{\Phi}_0, \quad \theta_k(0) = \tau_0, \quad c_k(0) = \Psi_0. \end{cases} \tag{16}$$

Passing to the limit in the linear terms of Eq. (16) is straightforward, and in the nonlinear terms $b(\vec{y}_k(t), \vec{y}_k(t), \vec{\Psi})$, $b_0(\vec{y}_k(t), \theta_k(t), \xi)$ and $b_0(\vec{y}_k(t), c_k(t), \eta)$ will be a direct consequence of Corollary 18. Then, we deduce that $(\vec{y}, \theta, c, u)$ is a variational solution of (2).

From the results of Section 2 we obtain that θ satisfies the energy estimate for the temperature. Moreover, the pass to the limit in the energy estimates for the velocity and the microorganisms concentration will be a direct consequence of Corollary 17. Thus, $(\vec{y}, \theta, c, u)$ is a weak solution of (2).

Finally, by the strong convergence of $\{c_k\}_{k\in\mathbb{N}}$ to c in $L^2(0,T;L^2(\Omega))$ (particularly in $L^1(A \times (0,T))$), we have

$$\frac{1}{\operatorname{meas}(A \times (0,T))} \int_{A\times(0,T)} c\,dx\,dt \leqslant \sigma.$$

So, the element $(\vec{y}, \theta, c, u) \in \mathcal{U}$. $\square$

Now, we can demonstrate the existence of optimal solutions for control problem (P):

Theorem 20. *The optimal control problem (P) has, at least, a solution.*

Proof. We consider a minimizing sequence $\{(\vec{y}_k, \theta_k, c_k, u_k)\}_{k\in\mathbb{N}} \subset \mathcal{U}$. Then, $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^2(\Gamma_T^1)$, which implies, thanks to estimates (4), that $\{\vec{y}_k\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;Y_0) \cap L^\infty(0,T;L^2(\Omega)^3)$, that $\{\theta_k\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;W_0) \cap L^\infty(0,T;L^2(\Omega))$, and that $\{c_k\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;W_1) \cap L^\infty(0,T;L^2(\Omega))$. Thus, we can take a subsequence of $\{(\vec{y}_k, \theta_k, c_k, u_k)\}_{k\in\mathbb{N}}$, still denoted in the same way, such that

$$\begin{aligned}&\{(\vec{y}_k, \theta_k, c_k, u_k)\} \rightharpoonup (\vec{y}_0, \theta_0, c_0, u_0)\\ &\quad \text{in } L^2(0,T;Y_0) \times L^2(0,T;W_0) \times L^2(0,T;W_1) \times L^2(\Gamma_T^1).\end{aligned}$$

From Lemma 19 we know that $\mathcal{U}$ is weakly closed. Thus, $(\vec{y}_0, \theta_0, c_0, u_0) \in \mathcal{U}$. Finally, due to the continuity and the convexity of the cost functional J (in particular, J is weakly lower semicontinuous), we deduce that

$$J(\vec{y}_0, \theta_0, c_0, u_0) \leqslant \liminf_{k\to\infty} J(\vec{y}_n, \theta_n, c_n, u_n) = \inf_{(\vec{y},\theta,c,u)\in\mathcal{U}} J(\vec{y}, \theta, c, u).$$

Thus, $(\vec{y}_0, \theta_0, c_0, u_0)$ is a minimum of J in $\mathcal{U}$. $\square$

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